Available online at www.sciencedirect.com
ScienceDirect
Stochastic Processes and their Applications 118 (2008) 1043–1055
stochastic processes and their applications
www.elsevier.com/locate/spa

Nonparametric estimation and testing time-homogeneity for processes with independent increments

Yoichi Nishiyama*

Department of Mathematical Analysis and Statistical Inference, The Institute of Statistical Mathematics, 4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan

Received 14 September 2006; received in revised form 13 April 2007; accepted 18 July 2007
Available online 26 July 2007

Abstract

We consider a nonparametric estimation problem for the Lévy measure of time-inhomogeneous process with independent increments. We derive the functional asymptotic normality and efficiency, in an ℓ^∞-space, of generalized Nelson–Aalen estimators. Also we propose some asymptotically distribution free tests for time-homogeneity of the Lévy measure. Our result is a fruit of the empirical process theory and the martingale theory.

MSC: Primary 62G05; 62G10; secondary 62N02; 62M02

Keywords: Empirical process; Process with independent increments; Lévy process; Martingale; Nonparametric estimation; Change point problem

1. Introduction

The importance of processes with independent increments in stochastic modelling has been well recognized. Indeed, they include Lévy processes and compound Poisson processes, which are nowadays known to be very useful in mathematical finance, as its special cases. Many researchers have investigated parametric statistical inference problems for Lévy processes in various situations. However, only a few papers have undertaken nonparametric approaches. Rubin and Tucker [12] constructed consistent estimators. Our work should be compared with

* Tel.: +81 354218726; fax: +81 354218796.
E-mail address: nisiyama@ism.ac.jp.

0304-4149/$ - see front matter
doi:10.1016/j.spa.2007.07.011

Basawa and Brockwell [3] who considered time-homogeneous non-decreasing Lévy processes. They proposed three estimators for weighted cumulative Lévy measure $u \rightsquigarrow \int_{(\varepsilon,u]}(z^2/(1+z^2))\alpha(\mathrm{d}z)$ on $[\varepsilon,\infty)$, and proved the functional asymptotic normality in $D[\varepsilon,\infty)$ for one of them, avoiding the discussion on jumps with sizes smaller than ε. For the rest two estimators they derived *some* functional asymptotic normality in $D[0,\infty)$ by putting $\varepsilon = 0$, but the limits were still some sequences of stochastic processes. To our best knowledge this problem is still open. We aim to fill this gap in the literature by applying a new theory of weak convergence established recently by Nishiyama [9,11]. The last paper which we should mention is [6], but their approach is essentially different from ours because it is based on discrete observation. Although we agree that discrete observation is more realistic, they have not yet obtained any functional asymptotic distribution theory.

Let $t \rightsquigarrow Z_t$ be a one-dimensional time-inhomogeneous process with independent increments, starting at Z_0, with the Lévy measure

$$L(\mathrm{d}t,\mathrm{d}z) = \mathrm{d}t\alpha(t,\mathrm{d}z).$$

Here, notice that $\int_{\mathbb{R}}\alpha(t,\mathrm{d}z)$ and $\int_{\mathbb{R}}|z|\alpha(t,\mathrm{d}z)$ are finite or infinite, and that $\int_{\mathbb{R}}(z^2\wedge 1)\alpha(t,\mathrm{d}z) < \infty$. Introducing a non-negative bounded measurable function w on $\mathbb{R}$, we aim to estimate

$$A(t,u) = \int_{[0,t]\times\mathbb{R}} w(z)1_{(-\infty,u]}(z)\mathrm{d}s\alpha(s,\mathrm{d}z), \quad (t,u)\in[0,1]\times\mathbb{R}.$$

Actually, we will consider a more general situation where the indicator functions $1_{(-\infty,u]}(z)$ are replaced by $\xi(z)$ from a class Ξ of functions on $\mathbb{R}$.

If Z is a compound Poisson process, then the weight function should be $w \equiv 1$. However, in our general setting the weight function given by

$$w(z) = z^2 \wedge 1 \quad \text{or} \quad \frac{z^2}{1+z^2}$$

is the most natural one. If we know that $\int_{\mathbb{R}}(|z|\wedge 1)\alpha(t,\mathrm{d}z) < \infty$ in advance, then the weight function should be $w(z) = |z|\wedge 1$.

Since it is not realistic to compute infinitely many sums, for the construction of the estimators and test statistics we will use the truncated weight function w_n defined by

$$w_n(z) = w(z)1_{\mathbb{R}\setminus[-c_n,c_n]}(z)$$

for a given sequence of non-negative constants $c_n \to 0$ as $n \to \infty$. (In the compound Poisson process case, we can set $c_n = 0$ for all $n \in \mathbb{N}$.) We emphasize that we *do* need the continuous observation of $t \rightsquigarrow Z_t$ but we do *not* use the jumps such that $|\Delta Z_t| \le c_n$ in the construction of our estimators and test statistics. We will prove the functional asymptotic normality and efficiency of proposed estimators for $(t,u) \rightsquigarrow A(t,u)$, in $\ell^\infty([0,1]\times\mathbb{R})$. Here, we denote by $\ell^\infty(T)$ the space of bounded functions defined on a set T, and equip it with the supremum norm $\|\cdot\|_\infty$.

The second topic considered in this paper is a nonparametric change point problem for Lévy measure. We wish to test the null hypothesis that the Lévy measure is time-homogeneous versus the alternative that there exists $t_0 > 0$ such that $\alpha(t,\mathrm{d}z) = \alpha_0(\mathrm{d}z)$ for $t \le t_0$ and $\alpha(t,\mathrm{d}z) = \alpha_1(\mathrm{d}z)$ for $t > t_0$, where $\alpha_0 \ne \alpha_1$. The corresponding results for the I.I.D. case can be found in [4, Section 2.6], and some authors have considered times series models. However, although Lee et al. [7] considered a parametric model of diffusion processes, there are few papers on nonparametric change point tests for continuous time stochastic processes. Our work seems to be new even for the compound Poisson processes.

The organization of this paper is as follows. In Section 2, we consider the general case of time-inhomogeneous processes with independent increments. The proposed estimator is a kind of Nelson–Aalen's estimator; see [1]. We extend the estimator for multivariate point processes studied by Nishiyama [8,10] to the case where the process may have infinitely many jumps (but we compute finite sums only!). We show the asymptotic normality and efficiency of proposed estimators. In Section 3, we consider the change point problem for (possibly) time-homogeneous processes with independent increments. We obtain some asymptotically distribution free tests under the null hypothesis. Also, we derive their asymptotic behavior under the alternatives. In the Appendix we state the result by Nishiyama [9,11] which is needed in the present work.

2. Estimation: Asymptotic normality and efficiency

Let a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$ be given. For every $k = 1, \ldots, n$, let $t \rightsquigarrow Z_t^k$ be a one-dimensional time-inhomogeneous Lévy process, starting at Z_0^k, with the Lévy measure

$$L(\mathrm{d}t, \mathrm{d}z) = \mathrm{d}t\alpha(t, \mathrm{d}z).$$

We assume that we can observe the process $t \rightsquigarrow Z_t^k$ only on a random time interval $[\sigma^k, \tau^k] \subset [0, 1]$, where σ^k and τ^k are stopping times. A typical example is that $Z_0^k > 0$, $\sigma^k = 0$ and $\tau^k = \inf\{t \geq 0 : Z_t^k \leq 0\} \wedge 1$. We can also consider the right censored case $\sigma^k = 0$ and $\tau^k = \inf\{t \geq 0 : Z_t^k \leq 0\} \wedge C^k \wedge 1$ where C^k is a stopping time which represents the censoring. However, our setting is more general; actually what we assume is just the condition (6) below. See also a remark after Theorem 1.

We introduce the integer-valued random measure N^k given by

$$N^k(\omega; \mathrm{d}t, \mathrm{d}z) = \sum_{s\in(\sigma^k(\omega),\tau^k(\omega)]} 1_{\{\Delta Z_s^k(\omega)\neq 0\}}\varepsilon_{(s,\Delta Z_s^k(\omega))}(\mathrm{d}t, \mathrm{d}z)$$

where ε_a denotes the Dirac measure at point a. Then, the predictable compensator of N^k is given by

$$1_{]]\sigma^k,\tau^k]]}(\omega, t)\mathrm{d}t\alpha(t, \mathrm{d}z),$$

where $]]\sigma, \tau]] = \{(\omega, t) : t \in [0, \infty), \sigma(\omega) < t \leq \tau(\omega)\}$. Summing up with respect to k, we have the integer-valued random measure

$$\mu^n(\omega; \mathrm{d}t, \mathrm{d}z) = \sum_{k=1}^{n} N^k(\omega; \mathrm{d}t, \mathrm{d}z). \tag{1}$$

To be an integer-valued random measure, it has to hold that $\mu^n(\{t\} \times \mathbb{R}) \leq 1$, so we assume that Z^k's are independent. (See Section II.1b of [5] for the theory of integer-valued random measures.) The predictable compensator of μ^n is given by

$$\nu^n(\omega; \mathrm{d}t, \mathrm{d}z) = Y_t^n(\omega)\mathrm{d}t\alpha(t, \mathrm{d}z) \tag{2}$$

where

$$Y_t^n(\omega) = \sum_{k=1}^{n} 1_{]]\sigma^k,\tau^k]]}(\omega, t).$$

We consider the estimation problem for α by using a kind of Nelson–Aalen estimator. A difference between [1,8,10] and the present work is that we assume $\alpha(t, \mathbb{R})$ may *not* be finite. Hence in our case, infinitely many jumps may occur. We analyze this problem via the martingale central limit theory of Nishiyama [11].

Let a countable class Ψ of measurable functions on $[0, 1] \times \mathbb{R}$ be given. Assume that $|\psi| \leq \bar{\psi}$ for every $\psi \in \Psi$ holds for a measurable function $\bar{\psi}$ on $[0, 1] \times \mathbb{R}$, which is called an *envelope function* for Ψ, such that

$$\int_{[0,1]\times\mathbb{R}} \bar{\psi}(t, z) \vee \bar{\psi}(t, z)^2 \mathrm{d}t\alpha(t, \mathrm{d}z) < \infty.$$

We actually estimate the functional

$$(t, \psi) \rightsquigarrow A(t, \psi) = \int_{[0,t]\times\mathbb{R}} \psi(s, z)\mathrm{d}s\alpha(s, \mathrm{d}z), \quad (t, \psi) \in [0, 1] \times \Psi.$$

The reason why the class Ψ is assumed to be countable is that our current situation corresponds to the Case B in [11]. However, in many cases we may assume that Ψ is countable without loss of generality; see a remark below.

We denote by $N_{[\,]}(\Psi, \|\cdot\|_{L^2(\alpha)}; \varepsilon)$ the *bracketing number* with respect to the semi-norm

$$\|\psi\|_{L^2(\alpha)} = \sqrt{\int_{[0,1]\times\mathbb{R}} |\psi(t, z)|^2 \mathrm{d}t\alpha(t, \mathrm{d}z)}.$$

(That is, $N(\varepsilon) = N_{[\,]}(\Psi, \|\cdot\|_{L^2(\alpha)}; \varepsilon)$ is the smallest integer $N(\varepsilon)$ such that there exist $N(\varepsilon)$ pairs (ψ_l^k, ψ_u^k), $k = 1, \ldots, N(\varepsilon)$ of elements of $L^2(\alpha)$ such that, for every $\psi \in \Psi$, $\psi_l^k \leq \psi \leq \psi_u^k$ holds for some k and that $\|\psi_l^k - \psi_u^k\|_{L^2(\alpha)} \leq \varepsilon$ for every k.) We will assume the *bracketing entropy condition*:

$$\int_0^1 \sqrt{\log N_{[\,]}(\Psi, \|\cdot\|_{L^2(\alpha)}; \varepsilon)}\mathrm{d}\varepsilon < \infty. \tag{3}$$

Here, let us discuss the choice of the class Ψ. Introducing a weight function w on $[0, 1] \times \mathbb{R}$ such that

$$\int_{[0,1]\times\mathbb{R}} |w(t, z)| \vee w(t, z)^2 \mathrm{d}t\alpha(t, \mathrm{d}z) < \infty, \tag{4}$$

we consider some classes like

$$\Psi = \{w_n(t, z)1_{(-\infty,u]}(z) : u \in \mathbb{R}\} \quad \text{or} \quad \{w_n(t, z)1_{(v,u]}(z) : -\infty < v \leq u < \infty\}, \tag{5}$$

etc., where $w_n(t, z) = w(t, z)1_{\mathbb{R}\setminus[-c_n,c_n]}(z)$ with $\{c_n\}$ being an appropriate sequence of non-negative constants such that $c_n \to 0$. Then, the condition (3) is satisfied, because we have $N_{[\,]}(\Psi, \|\cdot\|_{L^2(\alpha)}; \varepsilon) \leq K\varepsilon^{-1}$ for some constant $K > 0$.

Coming back to the general setting of the class Ψ, we propose a kind of Nelson–Aalen estimators $\widehat{A}^n$ given by

$$(t, \psi) \rightsquigarrow \widehat{A}^n(t, \psi) = \int_{[0,t]\times\mathbb{R}} \psi(s, z)1_{\mathbb{R}\setminus[-c_n,c_n]}(z)Y_s^{n-}\mu^n(\mathrm{d}s, \mathrm{d}z),$$

where

$$Y_s^{n-} = \begin{cases} 1/Y_s^n, & \text{if } Y_s^n \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. If the class Ψ is given by (5), then the process $(t, u) \rightsquigarrow \widehat{A}^n(t, u) - A(t, u)$ is separable. Hence, we may assume that u's are countable (e.g. rational numbers) without loss of generality.

The following is the main result of this section.

Theorem 1. *Suppose that the bracketing entropy condition (3) is satisfied. Suppose also that there exists a measurable function $y(t)$ on $[0, 1]$, which is bounded and bounded away from zero, such that*

$$\sup_{t\in[0,1]} |nY_t^{n-} - 1/y(t)| \longrightarrow 0 \quad \textit{in probability}, \ \textit{as } n \to \infty. \tag{6}$$

Suppose also that

$$\sqrt{n} \int_{[0,t]\times\mathbb{R}} \bar{\psi}(s, z) 1_{[-c_n,c_n]}(z) \mathrm{d}s\alpha(s, \mathrm{d}z) \to 0 \quad \textit{as } n \to \infty. \tag{7}$$

Then, it holds that $\sqrt{n}(\widehat{A}^n - A)$ converges weakly in $\ell^\infty([0, 1] \times \Psi)$, as $n \to \infty$, to the zero-mean Gaussian process G with the covariance

$$E(G(t, \psi)G(t', \psi')) = \int_{[0,t\wedge t']\times\mathbb{R}} \frac{\psi(s, z)\psi'(s, z)}{y(s)} \mathrm{d}s\alpha(s, \mathrm{d}z).$$

Furthermore, almost all paths of the process $(t, \psi) \rightsquigarrow G(t, \psi)$ are continuous with respect to the pseudo-metric ρ given by $\rho((t, \psi), (t', \psi')) = |t - t'| \vee \|\psi - \psi'\|_{L^2(\alpha)}$.

Remark. The assumption (6) has been a standard in the context of survival analysis; see (8.4.1) of [2]. See also (4.1.3) of [10].

Remark. Our framework includes the case where the process $t \rightsquigarrow Z_t$ is observed on $[0, T]$, and the Lévy measure L is periodic, that is,

$$L(\mathrm{d}t, \mathrm{d}z) = \sum_{k=1}^{\infty} 1_{(k-1,k]}(t)\mathrm{d}t\alpha(t - k + 1, \mathrm{d}z).$$

The above objects should be read as $Z_0^k = Z_{k-1}$ and $(\sigma^k, \tau^k] = (0, 1]$ for $k = 1, \ldots, [T]$, and $(\sigma^{[T]+1}, \tau^{[T]+1}] = (0, T - [T]]$. Then the assumption (6) is satisfied with $y(t) \equiv 1$ as $T \to \infty$.

Proof. Let us introduce the approximation $\widetilde{A}^n$ of A, given by

$$\widetilde{A}^n(t, \psi) = \int_{[0,t]\times\mathbb{R}} \psi(s, z) 1_{\mathbb{R}\setminus[-c_n,c_n]}(z) I_s^n \mathrm{d}s\alpha(s, \mathrm{d}z),$$

where $I_s^n = 1_{\{Y_s^n \geq 1\}}$. The assumption (6) implies that for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$P(Y_s^n Y_s^{n-} = 1 \text{ for all } s \in [0, 1]) \geq 1 - \varepsilon, \quad \forall n \geq n_0.$$

Recalling also (7), we have that $\sup_{t,\psi} \sqrt{n}|\widetilde{A}^n(t,\psi) - A(t,\psi)| \to 0$ in probability. Thus it is enough to analyze $\sqrt{n}(\widehat{A}^n - \widetilde{A}^n)$.

Here, notice that $X_t^{n,\psi} := \sqrt{n}(\widehat{A}^n(t,\psi) - \widetilde{A}^n(t,\psi)) = W^{n,\psi} * (\mu^n - \nu^n)_t$ where

$$W^{n,\psi}(\cdot, s, z) = \sqrt{n}\psi(s,z)1_{\mathbb{R}\setminus[-c_n,c_n]}(z)Y_s^{n-}.$$

Here $W*(\mu-\nu)$ denotes the stochastic integral (see Section II.1 of [5]). We will apply Corollary 2.7 of [11], with the help of Section 3 (especially, Theorem 3.4) of [9]. For every $\psi, \psi' \in \Psi$, since

$$\begin{aligned}
\langle X^{n,\psi}, X^{n,\psi'}\rangle_t &= (W^{n,\psi}W^{n,\psi'}) * \nu_t^n \\
&= n\int_{[0,t]\times\mathbb{R}} \psi(s,z)\psi'(s,z)|Y_s^{n-}|^2 Y_s^n \mathrm{d}s\alpha(s,\mathrm{d}z) \\
&= \int_{[0,t]\times\mathbb{R}} \psi(s,z)\psi'(s,z) n Y_s^{n-} \mathrm{d}s\alpha(s,\mathrm{d}z) \\
&\to \int_{[0,t]\times\mathbb{R}} \frac{\psi(s,z)\psi'(s,z)}{y(s)} \mathrm{d}s\alpha(s,\mathrm{d}z) \quad \text{in probability as } n\to\infty,
\end{aligned}$$

the condition [C2] in Section 3 of [9] holds. The condition [L2] also follows from a similar computation, noting the existence of the envelope function $\bar{\psi}$.

It remains to prove that there exists a decreasing series of finite partitions (DFP), which asymptotically separates Ψ, for which [PE] holds. Since $y(t)$ is bounded away from zero, the bracketing entropy condition (3) implies that there exist $N(\varepsilon)$ pairs (ψ_u^k, ψ_l^k), $k = 1, \ldots, N(\varepsilon)$ in $L^2(\alpha)$ which cover Ψ. Introduce a DPF Π of Ψ induced from these brackets, that is, $\Pi(\varepsilon) = \{\Psi(\varepsilon;k) : 1 \le k \le N(\varepsilon)\}$ is given by $\Psi(\varepsilon;k) = \{\psi \in \Psi : \psi_l^{\varepsilon,k} \le \psi \le \psi_u^{\varepsilon,k}\}$ with modification to make the partition disjoint. It is clear that this DFP asymptotically separates Ψ (see the Appendix). A similar computation as above yields that

$$|\sqrt{n}(\psi_l^{\varepsilon,k} - \psi_u^{\varepsilon,k})|^2 * \nu_1^n \le \varepsilon^2 \sup_{s\in[0,t]} |nY_s^{n-}|.$$

So by assumption (6) we obtain [PE].

As for the continuity of the limit, we first have the claim with respect to the pseudo-metric d_G given by

$$d_G((t,\psi),(t',\psi')) = \sqrt{E|G(t,\psi) - G(t',\psi')|^2}.$$

The pseudo-metric ρ is equivalent to d_G because $y(s)$ is bounded, thus the assertions follow. □

We finish this section with a brief discussion on the asymptotic efficiency of the proposed estimator, following the general theory developed in Chapter 3.11 of [13]. The approach is very similar to that in Section 4.1.2 of [10], so we only state the outline.

Denote

$$L^p = L^p([0,1]\times\mathbb{R}, \mathbf{B}([0,1])\otimes\mathbf{B}(\mathbb{R}), \frac{1}{y(t)}\mathrm{d}t\alpha(t,\mathrm{d}z)), \quad \forall p \ge 1.$$

We set $\mathbb{H} = L^2$ and $H = L^1 \cap L^\infty$. Consider a family $\mathbf{P}^n = \{P_h^n : h \in H\}$ of probability measures on $(\Omega, \mathcal{F})$ indexed by H given as follows: under the probability measure P_h^n, the Lévy measure of Z^k's is given by

$$\mathrm{d}t\alpha_h^n(t,\mathrm{d}z) = \left(1 + \frac{h(t,z)}{2\sqrt{n}y(t)}\right)^2 \mathrm{d}t\alpha(t,\mathrm{d}z).$$

Define μ^n by (1) and $\nu^{n,h}$ by its compensator under P_h^n, that is, (2) with α replaced by α_h^n. Since $H = L^1 \cap L^\infty \subset L^2$, if the filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in[0,1]}$ is self-exciting, the log-likelihood ratio is given by

$$\log \frac{\mathrm{d}P_h^n|\mathcal{F}_1}{\mathrm{d}P_0^n|\mathcal{F}_1} = \left(\log\left|1 + \frac{h}{2\sqrt{n}y}\right|^2\right) * \mu_1^n - \left(\left|1 + \frac{h}{2\sqrt{n}y}\right|^2 - 1\right) * \nu_1^{n,0}$$

(see, e.g., Theorem III.5.19 of [5]). So by the same way as in Proposition 4.1.3 of [10] we can show that this model is *locally asymptotically normal*.

Next we define the sequence of unknown parameter $A^n : H \to \ell^\infty([0,1]\times\Psi)$ by

$$A^n(h)(t,\psi) = \int_{[0,t]\times\mathbb{R}} \psi(s,z)\mathrm{d}s\alpha_h^n(s,\mathrm{d}z).$$

It is easy to see that this sequence A^n is *differentiable* with rate $\sqrt{n}$ and that its derivative $\dot{A} : H \to \ell^\infty([0,1]\times\Psi)$ is given by

$$\dot{A}(h)(t,\psi) = \langle f_{t,\psi}, h\rangle_{\mathbb{H}}, \quad \text{where} \quad f_{t,\psi}(s,z) = 1_{[0,t]}(s)\psi(s,z).$$

That is, we have

$$\sqrt{n}\|A^n(h) - A^n(0) - \dot{A}(h)\|_\infty \to 0 \quad \forall h \in H.$$

Finally, under this contiguous alternatives, we can show that the estimator $\widehat{A}^n$ is regular in the same way as in Proposition 4.1.4 of [10]. Also, the limit appearing in Theorem 1 achieves the bound of asymptotic efficiency in $\ell^\infty([0,1]\times\Psi)$. So we can conclude that our estimator is asymptotically efficient in the sense of the convolution theorem (Theorem 3.11.2 of [13]). The asymptotic efficiency in the sense of the asymptotic minimax theorem also holds for a certain choice of loss function. That is, for any bounded continuous subconvex loss function $\ell : \ell^\infty([0,1]\times\Psi) \to [0,\infty)$, it holds that

$$\sup_{I\subset H} \limsup_{n\to\infty} \sup_{h\in I} E_h^n \ell\left(\sqrt{n}(\widehat{A}^n - A^n(h))\right) = E\ell(G),$$

and that, for any estimator T^n,

$$\sup_{I\subset H} \liminf_{n\to\infty} \sup_{h\in I} E_h^n \ell\left(\sqrt{n}(T^n - A^n(h))\right) \geq E\ell(G),$$

where the supremum with respect to $I \subset H$ is taken over all finite subsets. See Theorem 3.11.5 of [13].

3. Testing time-homogeneity

Let a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,\infty)}, P)$ be given. Our starting point is the time-inhomogeneous case, that is, the Lévy measure L of the Lévy process $t \rightsquigarrow Z_t$ is given by

$$L(\mathrm{d}t,\mathrm{d}z) = \mathrm{d}t\alpha(t,\mathrm{d}z).$$

We assume that the process Z is observable on $[0, T]$, and consider the asymptotics $T \to \infty$.

We wish to test

H_0: the Lévy measure is time-homogeneous, that is, $\alpha(t, \mathrm{d}z) = \alpha_0(\mathrm{d}z)$, v.s.,

H_1: there exists $\theta_0 \in (0, 1)$ such that

$$\alpha(t, \mathrm{d}z) = \begin{cases} \alpha_0(\mathrm{d}z) & \text{for } t \in [0, \theta_0 T], \\ \alpha_1(\mathrm{d}z) & \text{for } t \in (\theta_0 T, T], \end{cases}$$

where $\alpha_0 \neq \alpha_1$.

Let us consider the class $\Psi = [0, 1] \times \Xi$, where Ξ is a class of measurable function on $\mathbb{R}$ with an envelope function $\bar{\xi}$ such that

$$\int_{\mathbb{R}} |\bar{\xi}(z)| \vee |\bar{\xi}(z)|^2 \alpha_0(\mathrm{d}z).$$

We define the semi-norm $\|\cdot\|_{L^2(\alpha_0)}$ by

$$\|\xi\|_{L^2(\alpha_0)} = \sqrt{\int_{\mathbb{R}} |\xi(z)|^2 \alpha_0(\mathrm{d}z)}.$$

We will discuss the choice of Ξ later.

We put

$$A^T(\theta, \xi) = \frac{1}{T} \int_{[0,T]\times\mathbb{R}} k_\theta(tT^{-1}) \xi(z) \mathrm{d}t \alpha(t, \mathrm{d}z), \quad \theta \in [0, 1], \ \xi \in \Xi,$$

where

$$k_\theta(t) = (1 - \theta) 1_{[0,\theta]}(t) - \theta 1_{(\theta,1]}(t).$$

Our idea comes from the fact that, under H_0, it holds that $A^T(\theta, \xi) = 0$ for all $(\theta, \xi) \in [0, 1] \times \Xi$; this is the essence of the results in this section. Under either H_0 or H_1, a natural estimator for A^T is

$$\widehat{A}^T(\theta, \xi) = \frac{1}{T} \int_{[0,T]\times\mathbb{R}} 1_{\mathbb{R}\setminus[-c_T, c_T]}(z) k_\theta(tT^{-1}) \xi(z) \mu(\mathrm{d}t, \mathrm{d}z),$$

where

$$\mu(\cdot; \mathrm{d}t, \mathrm{d}z) = \sum_s 1_{\{\Delta Z_s(\cdot) \neq 0\}} \varepsilon_{(s, \Delta Z_s(\cdot))}(\mathrm{d}t, \mathrm{d}z),$$

and where $\{c_T\}$ is a sequence of non-negative constants such that $c_T \to 0$.

Theorem 2. *Assume H_0. Suppose that*

$$\int_0^1 \sqrt{\log N_{[\,]}(\Xi, \|\cdot\|_{L^2(\alpha_0)}, \varepsilon)} \mathrm{d}\varepsilon < \infty. \tag{8}$$

Let $\{c_T\}$ be any sequence of non-negative constants such that $c_T \to 0$. Then, it holds that $\sqrt{T}\widehat{A}^T$ converges weakly in $\ell^\infty([0, 1] \times \Xi)$ to the zero-mean Gaussian process $(\theta, \xi) \rightsquigarrow G(\theta, \xi)$ with the covariance

$$EG(\theta, \xi)G(\theta', \xi') = (\theta \wedge \theta' - \theta\theta') \int_{\mathbb{R}} \xi(z)\xi'(z)\alpha_0(\mathrm{d}z).$$

Proof. Again, we use Corollary 2.7 of [11]. We define

$$\widetilde{A}^T(\theta,\xi) = \frac{1}{T}\int_{[0,T]\times\mathbb{R}} 1_{\mathbb{R}\setminus[-c_T,c_T]}(z)k_\theta(tT^{-1})\xi(z)\mathrm{d}t\alpha(t,\mathrm{d}z).$$

Then, under H_0, we have not only $A^T = 0$ but also $\widetilde{A}^T = 0$. Next observe that

$$\sqrt{T}(\widehat{A}^T(\theta,\xi) - \widetilde{A}^T(\theta,\xi)) = W^{T,\theta,\xi} * (\mu-\nu)_T,$$

where

$$W^{T,\theta,\xi}(t,z) = \frac{1}{\sqrt{T}} 1_{\mathbb{R}\setminus[-c_T,c_T]}(z)k_\theta(tT^{-1})\xi(z)$$

and $\nu(\mathrm{d}t,\mathrm{d}z) = \mathrm{d}t\alpha_0(\mathrm{d}z)$. Since

$$\begin{aligned}(W^{T,\theta,\xi}W^{T,\theta',\xi'}) * \nu_T &= \frac{1}{T}\int_{[0,T]\times\mathbb{R}} 1_{\mathbb{R}\setminus[-c_T,c_T]}(z)k_\theta(tT^{-1}) \\ &\quad \times \xi(z)k_{\theta'}(tT^{-1})\xi'(z)\mathrm{d}t\alpha_0(\mathrm{d}z) \\ &= \int_{[0,1]} k_\theta(s)k_{\theta'}(s)\mathrm{d}s \int_{\mathbb{R}} 1_{\mathbb{R}\setminus[-c_T,c_T]}(z)\xi(z)\xi'(z)\alpha_0(\mathrm{d}z) \\ &\to (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}} \xi(z)\xi'(z)\alpha_0(\mathrm{d}z) \quad \text{as } T\to\infty,\end{aligned}$$

we have the condition [C2] of [9]. His conditions [L2] and [PE] can be also easily checked. □

Based on this weak convergence result, by the continuous mapping theorem, we can derive the asymptotic behavior of the test statistics

$$\sup_{(\theta,\xi)\in[0,1]\times\Xi} |\sqrt{T}\widehat{A}^T(\theta,\xi)|. \tag{9}$$

We can choose any class Ξ which satisfies (8). The most elementary examples are as follows.

Corollary 3. *Choose w as in (4). Set*

$$\Xi_1 = \{w(z)1_{(-\infty,u]}(z) : u\in\mathbb{R}\},$$
$$\Xi_2 = \{w(z)1_{(v,u]}(z) : -\infty < v \le u < \infty\},$$

and define

$$\widehat{S}_i^T = \frac{\sup_{(\theta,\xi)\in[0,1]\times\Xi_i} |\sqrt{T}\widehat{A}^T(\theta,\xi)|}{\sqrt{4\widehat{A}^T(1/2,w^2)}}, \quad i=1,2.$$

Assume H_0, and suppose that $u \rightsquigarrow \int_{(-\infty,u]} |w(z)|^2\alpha_0(\mathrm{d}z)$ is continuous and that $\int_{\mathbb{R}} |w(z)|^2\alpha_0(\mathrm{d}z) > 0$. Then, $\widehat{S}_i^T$ converges weakly in $\mathbb{R}$ to S_i, where

$$S_1 = \sup_{t,s\in[0,1]} |B_t^\circ B_s|,$$
$$S_2 = \sup_{t,s,s'\in[0,1]} |B_t^\circ(B_s - B_{s'})|,$$

and where $t \rightsquigarrow B_t^\circ$ is a standard Brownian bridge and $t \rightsquigarrow B_t$ is a standard Brownian motion, which are independent.

Obviously, we can use more general classes Ξ like

$$\Xi_{2p} = \{w(z)1_{(v_1,u_1]\cup\cdots\cup(v_p,u_p]}(z) : -\infty < v_1 \leq u_1 \leq \cdots \leq v_p \leq u_p < \infty\}$$

and

$$\begin{aligned}\Xi_{2p}^{\pm} = \{&w(z)(1_{(v_1,u_1]\cup\cdots\cup(v_p,u_p]}(z) - 1_{\mathbb{R}\setminus(v_1,u_1]\cup\cdots\cup(v_p,u_p]}(z)) :\\ &-\infty < v_1 \leq u_1 \leq \cdots \leq v_p \leq u_p < \infty\},\end{aligned}$$

where p is a positive integer. The corresponding limits, namely S_{2p} and $S_{2p}^{\pm}$, are still distribution free.

Next, let us consider the asymptotic behavior of the test statistics under the alternative H_1.

Theorem 4. *Assume H_1. Suppose that*

$$\int_0^1 \sqrt{\log N_{[\,]}(\Xi, \|\cdot\|_{L^2(\alpha_0)}, \varepsilon)}\mathrm{d}\varepsilon + \int_0^1 \sqrt{\log N_{[\,]}(\Xi, \|\cdot\|_{L^2(\alpha_1)}, \varepsilon)}\mathrm{d}\varepsilon < \infty.$$

Let $\{c_T\}$ be a sequence of non-negative constants such that

$$\sqrt{T}\int_{\mathbb{R}} \bar{\xi}(z)1_{[-c_T,c_T]}(z)(\alpha_0(\mathrm{d}z) + \alpha_1(\mathrm{d}z)) \to 0 \quad \text{as } T \to \infty.$$

Then, it holds that $\sqrt{T}(\widehat{A}^T - A^*)$ converges weakly in $\ell^\infty([0,1]\times\Xi)$ to a zero-mean Gaussian process G^*, where

$$A^*(\theta,\xi) = \begin{cases} \theta(1-\theta_0)\int_{\mathbb{R}} \xi(z)(\alpha_0(\mathrm{d}z) - \alpha_1(\mathrm{d}z)), & \theta \in [0,\theta_0], \\ (1-\theta)\theta_0 \int_{\mathbb{R}} \xi(z)(\alpha_0(\mathrm{d}z) - \alpha_1(\mathrm{d}z)), & \theta \in (\theta_0, 1], \end{cases}$$

and the covariance of G^* is given by

$$\begin{aligned} &EG^*(\theta,\xi)G^*(\theta',\xi') \\ &= \begin{cases} (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}} \xi(z)\xi'(z)\alpha_0(\mathrm{d}z) & \\ \quad + \theta\theta'(1-\theta_0)\int_{\mathbb{R}} \xi(z)\xi'(z)(\alpha_1(\mathrm{d}z) - \alpha_0(\mathrm{d}z)), & \theta,\theta' \in [0,\theta_0], \\ (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}} \xi(z)\xi'(z)((1-\theta_0)\alpha_0(\mathrm{d}z) + \theta_0\alpha_1(\mathrm{d}z)), & \theta \in [0,\theta_0], \theta' \in [\theta_0, 1], \\ (\theta\wedge\theta' - \theta\theta')\int_{\mathbb{R}} \xi(z)\xi'(z)\alpha_1(\mathrm{d}z) & \\ \quad + (1-\theta)(1-\theta')\theta_0 \int_{\mathbb{R}} \xi(z)\xi'(z)(\alpha_0(\mathrm{d}z) - \alpha_1(\mathrm{d}z)), & \theta,\theta' \in [\theta_0, 1]. \end{cases} \end{aligned}$$

Proof. The proof is similar to that for Theorem 2, so it is omitted. □

Notice that the above theorem formally includes the case of the null hypothesis H_0; that is, the assumption $\alpha_0 \neq \alpha_1$ is not essential. In general, we have

$$\sqrt{T}\widehat{A}^T \approx \sqrt{T}A^* + G^* \quad \text{for large } T.$$

When $\alpha_0 = \alpha_1$, the deterministic process A^* is identically zero, and the Gaussian part becomes $G^* = G$ where G is from Theorem 2. When $\alpha_0 \neq \alpha_1$, the value $|A^*(\theta, \xi)|$ becomes strictly positive. So our test statistics (9) is consistent.

We mention that the alternative H_1 can be generalized into the following form:

H_1': $\alpha(t, \mathrm{d}z)$ depends on t, that is, there exists t_0 such that $\alpha(t, \mathrm{d}z) \neq \alpha(s, \mathrm{d}z)$ for $t \leq t_0 < s$.

Under this alternative H_1', the Lévy measure α may have plural change points, and such a case is more interesting in applications. However, the covariance structure of the limit process under the alternative becomes complicated.

The power of the test depends on the choice of Ξ, which is important in practice. For the sake of explanation, we shall consider the integrable case $\alpha(t, \mathrm{d}z) = \lambda(t)F(t, \mathrm{d}z)$ where $\lambda(t) < \infty$ and $F(t, \mathbb{R}) = 1$. When we treat the model

$$\alpha(t, \mathrm{d}z) = \lambda(t)F(\mathrm{d}z),$$

where F does not depend on t, the one element class $\Xi = \{1_{\mathbb{R}}(z)\}$ is trivially enough. On the other hand, when the model is

$$\alpha(t, \mathrm{d}z) = \lambda F(t, \mathrm{d}z),$$

where λ does not depend on t, the choice of the class Ξ is serious; if $F(t, \mathrm{d}z)$ and $F(s, \mathrm{d}z)$, where $t \leq t_0 < s$, are different in a complex way, then the class Ξ_{2p} or $\Xi_{2p}^{\pm}$ with a big integer p becomes powerful. The selection of p depends on the complexity of the model which is considered.

Acknowledgements

The author thanks Profs. S. Kuriki, A.W. van der Vaart, N. Yoshida and H. van Zanten for the discussions with them on the earlier versions of this article. He is also grateful to two referees for their constructive comments.

Appendix

We state here a result, which is needed in the present paper, based on [9–11].

Definition 5. Let $(\mathcal{X}, \mathcal{A}, \lambda)$ be a σ-finite measure space. For a given mapping $Z : \mathcal{X} \to \mathbb{R}\cup\{\infty\}$, we denote by $[Z]_{\mathcal{A},\lambda}$ any $\mathcal{A}$-measurable function $U : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ such that: (i) $U \geq Z$ holds identically; (ii) $\tilde{U} \geq U$ holds λ almost everywhere, for every $\mathcal{A}$-measurable function $\tilde{U}$ such that $\tilde{U} \geq Z$ holds λ almost everywhere.

The existence of such a random variable $[Z]_{\mathcal{A},\lambda}$ and its uniqueness up to a λ-negligible set follow from Lemma 1.2.1 of [13].

Definition 6. Let Ψ be an arbitrary set. $\Pi = \{\Pi(\varepsilon)\}_{\varepsilon\in(0,\Delta_\Pi]}$, where $\Delta_\Pi \in (0, \infty) \cap \mathbb{Q}$, is called a decreasing series of finite partitions (abb. DFP) of Ψ if it satisfies the following (i), (ii) and (iii): (i) each $\Pi(\varepsilon) = \{\Psi(\varepsilon; k) : 1 \leq k \leq N_\Pi(\varepsilon)\}$ is a finite partition of Ψ, that is, $\Psi = \bigcup_{k=1}^{N_\Pi(\varepsilon)} \Psi(\varepsilon; k)$; (ii) $N_\Pi(\Delta_\Pi) = 1$ and $\lim_{\varepsilon\downarrow 0} N_\Pi(\varepsilon) = \infty$; (iii) $N_\Pi(\varepsilon) \geq N_\Pi(\varepsilon')$ whenever $\varepsilon \leq \varepsilon'$.

Definition 7. We say a DFP $\Pi = \{\Pi(\varepsilon)\}_{\varepsilon\in(0,\Delta_\Pi]}$ of Ψ asymptotically separates Ψ if for any finite subset $F \subset \Psi$, there exists ε_F such that for every $\varepsilon \in (0, \varepsilon_F]$ each partitioning set $\Psi(\varepsilon; k)$ of the partition $\Pi(\varepsilon)$ contains at most one point of F.

This is not a strong requirement. In fact, consider the case where Ψ is totally bounded with respect to a metric ρ. When each $\Pi(\varepsilon)$ is a partition generated by ε-balls which cover Ψ, then $\Pi = \{\Pi(\varepsilon)\}_{\varepsilon\in(0,\Delta_\Pi]}$ asymptotically separates Ψ.

Let us now turn to the context of integer-valued random measure. Let $(E, \mathcal{E})$ be a Blackwell space. Let μ be an integer-valued random measure on $\mathbb{R}_+ \times E$ defined on a stochastic basis $\mathbf{B} = (\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, and ν a "good" version of the predictable compensator of μ. Let τ be a fixed time. We put $\tilde{\Omega} = \Omega \times \mathbb{R}_+ \times E$ and $\tilde{\mathcal{P}} = \mathcal{P} \otimes \mathcal{E}$ where $\mathcal{P}$ is the predictable σ-field. Let $\mathcal{W} = \{W^\psi : \psi \in \Psi\}$ be a family of predictable functions on $\tilde{\Omega}$ indexed by Ψ. We introduce the Doléans measure M_ν^P on $(\tilde{\Omega}, \tilde{\mathcal{P}})$, which is $\tilde{\mathcal{P}}$-σ-finite, given by

$$M_\nu^P(\mathrm{d}\omega, \mathrm{d}t, \mathrm{d}z) = P(\mathrm{d}\omega)\nu(\omega; \mathrm{d}t, \mathrm{d}z).$$

(See Section II.1 of [5] for the theory of random measures.)

Let us recall the definitions of the *predictable envelope* $\overline{W}$ and the *quadratic Π-modulus* $\|\mathcal{W}\|_\Pi$ given by [9,10]).

Definition 8. The predictable envelope $\overline{W}$ of $\mathcal{W} = \{W^\psi : \psi \in \Psi\}$ is defined by

$$\overline{W} = \left[\sup_{\psi\in\Psi} |W^\psi|\right]_{\tilde{\mathcal{P}}, M_\nu^P}.$$

For a given DFP Π of Ψ, the quadratic Π-modulus $\|\mathcal{W}\|_\Pi$ of $\mathcal{W} = \{W^\psi : \psi \in \Psi\}$ is defined as the $\mathbb{R}_+ \cup \{\infty\}$-valued predictable process $t \rightsquigarrow \|\mathcal{W}\|_{\Pi,t}$ given by

$$\|\mathcal{W}\|_{\Pi,t} = \sup_{\varepsilon\in(0,\Delta_\Pi]\cap\mathbb{Q}} \max_{1\le k\le N_\Pi(\varepsilon)} \frac{\sqrt{|\Delta_W(\Psi(\varepsilon;k))|^2 * \nu_t}}{\varepsilon} \qquad \forall t \in \mathbb{R}_+,$$

where

$$\Delta_W(\Psi') = \left[\sup_{\psi,\psi'\in\Psi'} |W^\psi - W^{\psi'}|\right]_{\tilde{\mathcal{P}}, M_\nu^P} \qquad \forall \Psi' \subset \Psi.$$

We will consider the following situation:

Case B in [11]: The process $t \rightsquigarrow (\overline{W}^2 \wedge \overline{W}) * \nu_t$ is locally integrable, Ψ is countable, and Π is a DFP which asymptotically separates Ψ.

Based on these preparations, we have the following claim.

Theorem 9 (*Corollary 2.7 of [11]*). *Let $(E, \mathcal{E})$ and Ψ be as that given above. For every $n \in \mathbb{N}$, let a stochastic basis $\mathbf{B}^n$ and the above objects μ^n, ν^n and $\mathcal{W}^n = \{W^{n,\psi} : \psi \in \Psi\}$ on $\mathbf{B}^n$ be given. Consider the above Case B. Suppose the following conditions:*

[PE] *there exists a DFP Π of Ψ such that*

$$\|\mathcal{W}^n\|_{\Pi,\tau} = O_{P^n}(1) \quad \text{and} \quad \int_0^{\Delta_\Pi} H_\Pi(\varepsilon)\mathrm{d}\varepsilon < \infty;$$

[L2] $|\overline{W}^n|^2 1_{\{\overline{W}^n > \varepsilon\}} * \nu_\tau^n \xrightarrow{P^n} 0$ *for every $\varepsilon > 0$;*

[C2] $\langle X^{n,\psi}, X^{n,\psi'}\rangle_t \xrightarrow{P^n} C_t^{(\psi,\psi')}$ *for every $t \in [0, \tau]$ and $(\psi, \psi') \in \Psi^2$, where the family $\{C_t^{(\psi,\psi')} : t \in [0, \tau], (\psi, \psi') \in \Psi^2\}$ of constants satisfies that*

$$t \rightsquigarrow C_t^{(\psi,\psi')} \text{ is continuous for every } (\psi, \psi') \in \Psi^2.$$

Then, the stochastic process $(t, \psi) \rightsquigarrow X_t^{n,\psi}$ *defined by*

$$X_t^{n,\psi} = W^{n,\psi} * (\mu^n - \nu^n)_t$$

converges weakly in $\ell^\infty([0, \tau] \times \Psi)$ *to a zero-mean Gaussian process* $(t, \psi) \rightsquigarrow G(t, \psi)$ *with the covariance* $EG(t, \psi)G(t', \psi') = C_{t\wedge t'}^{(\psi,\psi')}$. *Moreover, almost all paths of the limit process are uniformly* ρ*-continuous where*

$$\rho((t, \psi), (t', \psi')) = \sqrt{C_t^{(\psi,\psi)} + C_{t'}^{(\psi',\psi')} - 2C_{t\wedge t'}^{(\psi,\psi')}}.$$

Remark. It is possible to replace the fixed time τ by a stopping time τ^n. In this case, the convergence should be considered not in $\ell^\infty([0, \tau] \times \Psi)$ but in $\ell^\infty(\Psi)$.

References

[1] O.O. Aalen, Nonparametric inference for a family of counting processes, Ann. Statist. 6 (1978) 701–726.

[2] P.K. Andersen, Ø. Borgan, R.D. Gill, N. Keiding, Statistical Models Based on Counting Processes, Springer-Verlag, New York, 1993.

[3] I.V. Basawa, P.J. Brockwell, Non-parametric estimation for non-decreasing Lévy processes, J. Roy. Statist. Soc. B 44 (1982) 262–269.

[4] M. Csörgő, L. Horváth, Limit Theorems in Change-Point Analysis, Wiley, New York, 1997.

[5] J. Jacod, A.N. Shiryaev, Limit Theorems for Stochastic Processes, Springer-Verlag, Berlin, Heidelberg, 1987.

[6] G. Jongbloed, F.H. van der Meulen, A.W. van der Vaart, Nonparametric inference for Lévy driven Ornstein–Uhlenbeck processes, Bernoulli 11 (2005) 759–791.

[7] S. Lee, Y. Nishiyama, N. Yoshida, Test for parameter change in diffusion processes by cusum statistics based on one-step estimators, Ann. Inst. Statist. Math. 58 (2006) 211–222.

[8] Y. Nishiyama, Some central limit theorems for ℓ^∞-valued semimartingales and their applications, Probab. Theory Related Fields 108 (1997) 459–494.

[9] Y. Nishiyama, Weak convergence of some classes of martingales with jumps, Ann. Probab. 28 (2000) 685–712.

[10] Y. Nishiyama, Entropy methods for martingales, in: CWI Tract, 128, Centrum voor Wiskunde en Informatica, Amsterdam, 2000.

[11] Y. Nishiyama, On the paper “Weak convergence of some classes of martingales with jumps”, Ann. Probab. 35 (2007) 1194–1200.

[12] H. Rubin, H.G. Tucker, Estimating the parameters of a differential process, Ann. Math. Statist. 30 (1959) 641–658.

[13] A.W. van der Vaart, J.A. Wellner, Weak Convergence and Empirical Processes: With Applications to Statistics, Springer, New York, 1996.